1 Introduction

You will learn enough about complex numbers in this article to complete the programs `Complex.java` and `CMath.java`. No specific knowledge of complex arithmetic or complex functions is assumed. Everything will be done from scratch.

You are assumed to know about real numbers and their arithmetic. We will realize the complex numbers as an extension of the real numbers. They will have the four basic operations of arithmetic, addition, subtraction, multiplication, and division.

2 Extension and Number Systems

When you first learned about numbers from Mrs. Wormwood, you learned about the *natural numbers* $\mathbb{N}$; these are the positive counting numbers $\{1, 2, 3, 4, ...\}$. You learned about the operation of addition, first by using your fingers, then by using the efficient *ripple-carry* algorithm which looks like this.

```
1 1
 6322
+4382
 ----
10704
```

Also, you learned how to subtract natural numbers first by using your fingers, then by using the "borrowing" method. For quite some time, you were steered around mysteries such as $3 - 8$. At some juncture, you were introduced to negative numbers and you saw that $3-8 = -5$. This gave you signed arithmetic and introduced you to the set of all integers

$$\mathbb{Z} = \{\cdots - 2, -1, 0, 1, 2, \dots\}.$$

The integers behave nicely with $+$, - and *: the product, difference and sum of integers is yet another integer. This property is called *closure*; the integers are closed under $+$, - and * becasue sums, products and differences of integers are integers. More generally, if f is a polynomial with integer coefficients and $a \in \mathbb{Z}$, $f(a) \in \mathbb{Z}$.

Unfortunately, the integers are not closed under exact division, although they are closed under integer division. This brings us to and extension of the integers, the *rational numbers*, which are known by the symbol $\mathbb{Q}$, for "quotient."

The rules for arithmetic were then extended again by Mrs Wormwood to work for fractions. You have closure under division in the rationals, save for the special case of 0; this number cannot be a divisor of anything but itself. With the rationals, you can solve any equation of the form

$$ax + b = 0,$$

provided that $a \neq 0$.. The result, obviously, is

$$x = \frac{a}{b}.$$

Now comes geometry to break our hearts. Start with a square that is 1×1. The area of this square is 1. Now make a 2×2 array of these; you have four squares looking like this.

```
------A------
|     |     |
|     |     |
D-----+-----B
|     |     |
|     |     |
------C------
```

You think you see 5 squares in the picture, right Internet smarty? There is another that can be found connecting the points A, B, C and D labeled in the array of squares in the figure. You will notice that this square must have area 2. Let s be the length of one of its sides. We have $s^2 = 2$.

Here is an interesting question: *Can we find integers p and q so that*

$$\frac{p}{q} = \sqrt{2}?$$

Let us assume we can. Note that we can always write such a fraction in a fully reduced form by dividing the top and bottom by $\gcd(p, q)$. So we will assume it is fully reduced.

Squaring, we get

$$\frac{p^2}{q^2} = 2,$$

so

$$p^2 = 2q^2.$$

We are going to use the following result. If P is a prime number and P divides ab evenly, then P must divide on of a or b evenly.

Since $p^2 = 2q^2$, we see that p^2 is even. But the number 2 is prime so if 2 divides p^2, 2 must divide p. Hence p is even, and 4 divides p^2 evenly.

Now p^2 is divisible by 4, and $p^2 = 2q^2$, so $2q^2$ is divisible by 4. this says that q^2 is even, forcing q to be even.

We see that both p and q are even. But we assumed at the outset that they had no common factors. This is a contradiction. Our original premise, that $\sqrt{2}$ is rational, is false.

The rationals are not rich enough to account for all distances along a line. How do we get out of this jam? Again, we wish to extend our notion of number.

What we would like is for the real numbers to be realizable as a geometric line. This is certainly true for $\sqrt{2}$. You can construct this length with a compass and ruler. Just make two perpendicular lines and mark off one unit from the intersection. Now draw the square described by this process; its diagonal by the Pythagorean theorem is of length $\sqrt{2}$. It is not difficult to see that you can also construct any rational number in the line with straight edge and compass. This process of completion to a line gives us the real number $\mathbb{R}$. The actual construction of the real numbers is a hairy complex process. But it gives the real numbers the properties that make calculus go and it enables us to represent the real numbers as a line.

This brings us to the doorstep of another mystery. What about this equation?

$$x^2 + 1 = 0.$$

We know that if we square any real number, the result is nonnegative, so

$$x^2 + 1 \geq 1,$$

for all real numbers x. It could be handy to have an extension of the real numbers so this equation has a solution. Mathematically we say the real numbers are not *algebraically complete*; there are polynomials with real coefficients lacking real roots, such as the one we have exhibited here.

NC-17, For Afficianadoes of Power Series The geometric series theorem says that if x is a real number and if $x \neq 1$, then

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1}.$$

Now suppose that $|x| < 1$. Then $x^n \to 0$ as $n \to \infty$. Taking the limit on both sides of the previous equation yields

$$\sum_{n=0}^{\infty} x^n = \frac{-1}{x - 1} = \frac{1}{1 - x}.$$

So far all looks innocent. However, we can perform a little substitution of $-x^2$ for x to get

$$\sum_{n=0}^{\infty} x^{2n} = \frac{1}{1 + x^2} \qquad |x| < 1.$$

This works; you can do a little programming experiment in Python to see it. Draw the graph of the function $x \mapsto (1/(1+x^2))$. It is a nice bell-shaped affair. Nothing blows up, but the series for some mysterious reason behaves badly if $|x| \geq 1$. Exactly what sort of chicanery is going on here? Things like this happen for reasons!

Whatever could be lurking? Take some derivatives and see that

$$f'(x) = -\frac{2x}{(1+x^2)^2}$$

and

$$f''(x) = \frac{1-3x^2}{(1+x^2)^3}.$$

These functions are well mannered, and the denominators are bounded below by 1, so this trail offers nary a hint. Uh oh. Dead end. Perhaps if we could extend the real numbers in such a way that this polynomial has a root, the mystery might reveal itself.

3 A New Number System

We are going to extend the real numbers with a new set of objects called *complex numbers.* A complex number is a symbol of the form $a + bi$, where $a, b \in \mathbb{R}$; the totality of these objects is denoted by $\mathbb{C}$. We are going to define the operations of arithmetic on these objects. You will program in in Java and create this number system as a new data type.

If a and b are real, and $z = a + bi$, we say that a is the *real part* of z and denote it by $\Re(z)$, and that b is the *imaginary part* of z and denote it by $\Im(z)$. We say two complex numbers are equal if their real parts and imaginary parts are equal.

3.1 The Plane Facts

It is easy and useful to represent the complex numbers as points in the plane. If $z = a + bi$, and $a, b \in \mathbb{R}$, we shall represent it as the point (a, b) in the plane. In this plane, the x–axis is is the set of all complex numbers whose imaginary parts are zero, and the y-axis is the set of all "purely imaginary" numbers.

Every real number a can be thought of as complex number by writing $a + i0$. We will make this association without further remark. In particular, the zero for addition is $0 + 0i$, which we will henceforth know as 0. This is exactly what happens when we think of the x–axis in the complex plane as being a copy of the real numbers.

For this reason the x–axis also called the *real axis*. Likewise, the y–axis is called the *imaginary axis*.

As we develop the complex numbers, this planar representation will bring some interesting geometric meaning to addition and multiplication. This geometric relationship will be key to solving the mystery of the stopping power series.

3.2 Adding Complex Numbers

We begin by defining addition; to do so we insist that non-i terms and i terms are unlike and we add them algebraically. For example,

$$(1 + 4i) + (5 - 2i) = 6 + 2i.$$

It is not difficult to check that this operation is commutative and associative.

If you have seen vectors before, this form of addition is just vector addition. You can think of a complex number as a vector in the plane.

If you have not seen vectors before, here are the essentials. A vector is a directed line segment, which simply means it is a line segment with labeled ends. The best way to think of this is that a vector is an arrow with a head and a tail. The complex number $z = a + bi$ can be thought of as an arrow with its tail at the origin and its head at the point (a, b). This vector represents displacement by a units horizontally and b units vertically. If you look in a Physics book, you will see this vector denoted by $ai + bj$, where i represents one horizontal unit of displacement and j represents one vertical unit of displacement.

Every complex number $a + bi$ has an *additive inverse* which is $-a - bi$. If $z = a + bi$, then we will write $-z$ for $-a - bi$. You will notice that if your think of z as a vector, then $-z$ is just the vector pointing in the opposite direction of z with the same length.

Subtraction is defined as follows. If $z, w \in \mathbb{C}$, then we define

$$z - w = z + (-w).$$

Happily, this enterprise just boils down to subtracting and combining like terms algebraically. Be careful when computing $z - w$ to distribute the $-$ to both terms of w.

3.3 Addition and Geometry

Suppose that $z, w \in \mathbb{C}$. Think of them as vectors with their tails at the origin. You can see that they span a parallelogram. (Draw one!) Now imagine taking the tail of z and dragging it to the head of w. This will form one side of the spanning parallelogram. Now do the reverse: drag the tail of w and stick it on the head of z. The result of doing either puts the head of the second vector at the same point! Draw a new vector from the origin to this point; this is $z + w$. Notice how this supplies a geometric interpretation to addition. This is what is commonly known as the *parallelogram law.*

3.4 Argument

If $z \in \mathbb{C}$ is nonzero, we define the *argument* of z, $\text{Arg}(z)$ to be the angle in $(-\pi, \pi]$ between the real axis and z created when z is realized as a vector, and placed at the origin. For example,

$$\text{Arg}(1 + i) = \pi/4$$

since $1+i$ is one-eighth of a circle from from the real axis in the counterclockwise direction. Likewise,

$$\text{Arg}(1 - i) = -\pi/4.$$

Observe that for any positive real number z in the complex plane, $\text{Arg}(z) = 0$ and if z is real and negative, $\text{Arg}(z) = \pi$.

3.5 Multiplication, Conjugation, and Integer Powers

Now comes multiplication. We will first define $i * i = -1$. Now we shall extend this to any complex numbers by saying that the distributive law (and therefore the FOIL rule) works. This suffices. Notice what happens when we multiply two complex numbers. If $z = a + bi$ and $w = c + di$, then

$$zw = (a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + i(ad + bc).$$

Multiplication is commutative, associative and enjoys the distributive property. Its neutral element is $1 = 1 + 0i$.

Finally, we must determine if every complex number has a multiplicative inverse. To this end, we first make a helpful definition. If $z = a + bi$, then we define the *conjugate* of z by $\overline{z} = a - bi$. Observe that in this case,

$$z\overline{z} = (a + bi)(a - bi) = a^2 - b^2 i^2 = a^2 + b^2.$$

You should notice that $\sqrt{z\overline{z}}$ is the planar distance from z to the origin.

We will therefore define the *absolute value* of a complex number z by $|z| = \sqrt{z\overline{z}}$. This is the distance from the origin to the complex number. We then define the *distance* between $z, w \in \mathbb{C}$ by

$$d(z, w) = |z - w|.$$

This is just the planar distance between z and w.

3.6 Division

Let us begin by seeing that any nonzero complex number has a multiplicative inverse. Let $z \in \mathbb{C}$ be nonzero. Then $|z| > 0$. We will now show that

$$w = \frac{\overline{z}}{|z|^2}$$

is a multiplicative inverse for z. To see this, just multiply as follows

$$zw = z \cdot \frac{\overline{z}}{|z|^2} = \frac{z\overline{z}}{|z|^2} = 1.$$

Now suppose $z = a + bi$. Then

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - i \cdot \frac{b}{a^2 + b^2}.$$

Since $z \neq 0$, you can be assured that $a^2 + b^2 > 0$, so the expression has no possibility of a division by zero.

We can now define positive integer powers just as you might expect, and we can define for any positive integer n,

$$z^{-n} = \frac{1}{z^n} = \left(\frac{1}{z}\right)^n.$$

Now, all of the rational functions we know from the real domain can be extended to the complex numbers. Let us now return to the mysterious stoppage of convergence we saw before.

3.7 Mystery Solved

We saw that

$$\frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \qquad -1 < x < 1.$$

The geometric series theorem works in the complex case just as for the real one; to wit

$$\frac{1}{1 + z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}, \qquad |z| < 1.$$

Next observe that $z^2 + 1 = (z - i)(z + i)$; this tells us that the rational function $z \mapsto 1/(z^2 + 1)$ blows up at $z = \pm i$. It is no coincidence that the real version on the series stops converging for x with $|x| \geq 1$. The distance from $\pm i$ to the origin is 1. This series actually converges in the biggest domain possible for it.

3.8 Is another extension needed here?

In short, no. The fundamental theorem of algebra states that all complex polynomials factor into linear factors. This proof of this result requires complex calculus and the Liouville Theorem.

4 Complex Functions

The most important complex function outside of the rational functions is the exponential function, which is defined by

$$e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad z \in \mathbb{C}.$$

This infinite sum converges for any complex number since the terms go to zero faster than any exponential function of n, no matter the choice of z.

There is an interesting cyclic pattern for powers of i which will now come into play. We can easily see that $i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$. The powers of i repeat cyclically every 4th power.

We have that $i^{2k} = (-1)^k$ and $i^{2k+1} = i^{2k}i = (-1)^k i$. This works for any integer k.

Suppose that $t \in \mathbb{R}$. Then we have

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \sum_{n=0}^{\infty} \frac{(it)^{2k}}{(2k)!} + \sum_{n=0}^{\infty} \frac{(it)^{2k+1}}{(2k+1)!}$$

This holds because we sum the even and odd terms separately. Now let us take advantage of what we know about powers of i to see that

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^{2k}}{(2k)!} + \sum_{n=0}^{\infty} \frac{(it)^{2k+1}}{(2k+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^k}{(2k)!} + i \sum_{n=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!}.$$

This is the famous Euler Theorem, which says

$$e^{it} = \cos(t) + i\sin(t).$$

Hence, if $z = a + ib$,

$$e^z = e^{a+ib} = e^a e^{ib} = e^a(\cos(b) + i\sin(b)).$$

From this the pair of relationships

$$e^{iz} = \cos(z) + i\sin(z)$$

and

$$e^{-iz} = \cos(-z) + i\sin(-z) - \cos(z) - i\sin(z).$$

Now add and you can see that

$$2\cos(z) = e^{iz} + e^{-iz},$$

so that

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}.$$

Likewise, if you subtract you see that

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

4.1 Some Consequences

In the real domain, we know that the exponential function is increasing, and it is therefore 1-1. It has an inverse, called the (natural) log function. In the complex domain, we have the following.

$$e^{z+2i\pi} = e^z e^{2i\pi} = e^z(\cos(2\pi) + i\sin(2\pi) = e^z.$$

The complex exponential function is periodic with period $2i\pi$. It is not 1-1 so there is not a cleanly defined log function on the complex plane.

It is also interesting to notice that for any $t \in \mathbb{R}$,

$$|e^{it}|^2 = \cos^2(t) + \sin^2(t) = 1.$$

The function $t \mapsto e^{it}$ maps the real line onto the unit circle by sketching it out counterclockwise. This function has a period of 2π, so the interval $(-\pi, \pi]$ constitutes one trip around the unit circle.

As a result every complex number z can be written as

$$z = re^{it},$$

where $r = |z|$ and $t \in (pi, \pi]$. If $z \neq 0$, we have $r > 0$ so we can write

$$z = e^{\log(r)} + e^{it} = e^{\log(r)+it}.$$

The quantity $\log(r) + it$ is called the *principal branch of the logarithm* in the complex plane. You should also notice that in this case, $t = \mathrm{Arg}(z)$. We henceforth define

$$mathrmLog(z) = \log|z| + i\mathrm{Arg}(z).$$

If $w \in \mathbb{C}$, then we can define

$$z^w = e^{w\mathrm{Log}(z)}.$$

5 The Complex Versions of the Trigonometric Functions

We see that for a real number t, we have

$$\cos(t) = \frac{e^{it} + e^{-it}}{2}$$

and

$$\sin(t) = \frac{e^{it} - e^{-it}}{2i}.$$

We can therefore define for $z \in \mathbb{C}$

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

and

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$