Sequences

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0 Introduction

The purpose of this article is to introduce the reader to the mathematical language ancillary to sequences. This includes various mathmatical operations on sequences. We will use the symbol $\mathbb{N}_0$ to denote the nonnegative integers $\{0, 1, 2, 3, ...\}$, and the symbol $\mathbb{R}$ to denote the real numbers.

1 A Ton of Defintions

To get started, let's define some terms.

Definition. A *sequence* is a function $a : \mathbb{N}_0 \to \mathbb{R}$. We will use the traditional notation $a_n = a(n)$. The number n is the *index* of the sequence a at n.

You should think of the indices of a sequence as living between the entries; each index points to the entry just to its right.

Definition. The *zero sequence* 0 is defined by $0_n = 0$ for all $n \in \mathbb{N}_0$. Sequences that are constant functions are called *constant sequences.*

Algebraic operations on sequences are defined in the canonical way. If a and b are sequences, we define the following

1. $(a + b)_n = a_n + b_n$, $n \in \mathbb{N}_0$
2. $(a - b)_n = a_n - b_n$, $n \in \mathbb{N}_0$
3. $(ab)_n = a_n b_n$, $n \in \mathbb{N}_0$
4. $(a/b)_n = a_n/b_n$, $n \in \mathbb{N}_0$

It would be accurate to describe these definitions as being "entrywise operations."

Notice that if a is a sequence and $f : \mathbb{R} \to \mathbb{R}$, then the composition $f \circ a : \mathbb{N}_0 \to \mathbb{R}$ is defined. We define

$$f(a)_n = f(a_n), \qquad n \in \mathbb{N}_0.$$

The *convolution* of sequences a and b is defined as

$$a * b(n) = \sum_{k=0}^{n} a_k b_{n-k}.$$

2 Convolution Examples

Let's compute an example Consider the sequences $a_n = 1/2^n$ and $b_n - 1/3^n$, $n \in \mathbb{N}_0$. In our computation we will use the geometric series theorem which says that

$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}, \qquad z! = 1.$$

For $n \in \mathbb{N}_0$,

$$\begin{aligned}
(a * b)_n &= \sum_{k=0}^{n} a_k b_{n-k} \\
&= \sum_{k=0}^{n} \frac{1}{2^k} \frac{1}{3^{n-k}} \\
&= \frac{1}{3^n} \sum_{k=0}^{n} \left(\frac{3}{2}\right)^k \\
&= \frac{1}{3^n} \frac{1 - (3/2)^{n+1}}{1 - 3/2} \\
&= \frac{2}{3^n} \big((3/2)^{n+1} - 1\big) \\
&= \frac{3}{2^n} - \frac{2}{3^n}.
\end{aligned}$$

This computation is easily generalizable.

Here is another example consider the sequence $a_n = n$ for $n \in \mathbb{N}_0$. We will use these two facts that hold for $n \in \mathbb{N}_0$:

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

and

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Then

$$\begin{aligned}
(a * a)_n &= \sum_{k=0}^{n} a_k a_{n-k} \\
&= \sum_{k=0}^{n} k(n-k) \\
&= \sum_{k=0}^{n} nk - k^2 \\
&= n \sum_{k=0}^{n} k - \sum_{k=0}^{n} k^2 \\
&= \frac{n^2(n+1)}{2} + \frac{n(n+1)(2n+1)}{6} \\
&= n(n+1)\left(\frac{n}{2} - \frac{(2n+1)}{6}\right) \\
&= \frac{1}{6} n(n+1)(n-1)
\end{aligned}$$