AVERAGING FUNCTIONS

In this lab, we shall study the notion of the average value of a function and figure out what the definition should be. Let us do this for a simple function, $f(x) = x^2$ on the interval $[0, 2]$.

1. Compute an approximate average value for f as follows.

$$a_1 0(f) = \frac{f(0) + f(.2) + f(.4) + \cdots + f(2.0)}{11}.$$

What do you get? Why 11 and not 10? We have computed an approximate average value by using 11 equally spaced *sample points.*

2. Compute the approximate average value for f with 100 sample points, $a_{100}(f)$.

3. Using summation notation, write a formula for $a_n(f)$. What is a general formula for this for an arbitrary function f defined on a closed, bounded interval $[a, b]$?

4. Let's go back to the case of $f(x) = x^2$, $x \in [0, 2]$. Recall the identity

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \qquad n \geq 1.$$

Use this identity to obtain a closed–form expression for $a_n(f)$.

5. Let $n \to \infty$ in the last part of this problem. What should the average value of f be on $[0, 2]$?

6. Let g be a constant function defined on $[0, 2]$. Pick its (constant) value so that

$$\int_0^2 g(x)\, dx = \int_0^2 f(x)\, dx.$$

7. Let f be a function defined on a closed bounded interval $[a, b]$. Find a constant function g so that

$$\int_a^b f(x)\, dx = \int_a^b g(x)\, dx.$$

We are now going to assume our function is continuous. Note the following result

Theorem. *Let f be a continuous function on an interval $[a, b]$. If $f \geq 0$ and if $\int_a^b f(x)\, dx = 0$ then $f = 0$ on $[a, b]$.*

8. Show that if f is a continuous nonnegative function on a closed bounded interval $[a, b]$ with $a < b$ with zero average value, it must be zero. Is the hypothesis $a < b$ really necessary?

9. Suppose that f is a continuous function on the interval $[0 \infty]$. We define the *average value* of f to be

$$a(f) = \lim_{T \to \infty} \int_0^T f(t)\, dt.$$

Compute the average value of the function $f(x) = e^{-x}$, $x \geq 0$. Comment on it in light of the last two results.

10. Compute the average values of $x \mapsto \sin(x)$ and $x \mapsto \sin^2(x)$ on $[0, \infty)$. What can you say about a sinusiod of the form

$$a + b\cos(\lambda x) + c\sin(\mu x),$$

where λ and μ are nonzero constants, and a, b and c are arbitrary real constants?